THE CROSSING NUMBER OF COMPOSITE KNOTS

MARC LACKENBY

1. Introduction

One of the most basic questions in knot theory remains unresolved: is crossing number additive under connected sum? In other words, does the equality $c(K_1 \sharp K_2) = c(K_1) + c(K_2)$ always hold, where $c(K)$ denotes the crossing number of a knot K and $K_1 \sharp K_2$ is the connected sum of two (oriented) knots K_1 and K_2? The inequality $c(K_1 \sharp K_2) \leq c(K_1) + c(K_2)$ is trivial, but very little more is known in general. Equality has been established for certain classes of knots, most notably when K_1 and K_2 are both alternating ([3], [6], [7]) and when K_1 and K_2 are both torus knots [1]. In this paper, we provide the first non-trivial lower bound on $c(K_1 \sharp K_2)$ that applies to all knots K_1 and K_2.

Theorem 1.1. *Let $K_1, \dots, K_n$ be oriented knots in the 3-sphere. Then*

$$\frac{c(K_1) + \dots + c(K_n)}{152} \leq c(K_1 \sharp \dots \sharp K_n) \leq c(K_1) + \dots + c(K_n).$$

More generally, one can speculate about the crossing number of satellite knots. Here, there are a variety of conjectures, all of which remain wide open at present. The simplest of these asserts that the crossing number of a non-trivial satellite knot is at least the crossing number of its companion. To explain this, we fix some terminology. A knot K is a *non-trivial satellite knot with companion knot* L if K lies in a regular neighbourhood $N(L)$ of the non-trivial knot L, and K does not lie in a 3-ball contained in $N(L)$, and K is not a core curve of the solid torus $N(L)$. In a forthcoming article [4], we will prove the following result, by generalising the methods in this paper.

Theorem 1.2. *There is a universal computable constant $N \geq 1$ with the following property. Let K be a non-trivial satellite knot, with companion knot L. Then $c(K) \geq c(L)/N$.*

Here, 'universal' means that N is just a number, and 'computable' means that we have an algorithm to determine it. However, the constant N is more difficult to calculate than in the case of composite knots. We hope to find an explicit upper bound on N, but it will probably be significantly bigger than 152.

Let us start with an outline of the proof of Theorem 1.1. Let $K_1, \ldots, K_n$ be a collection of oriented knots. Our aim is to show that $c(K_1) + \ldots + c(K_n) \leq 152\, c(K_1 \sharp \ldots \sharp K_n)$. It is not hard to show that we may assume that each K_i is prime and non-trivial. Let D be a diagram of $K_1 \sharp \ldots \sharp K_n$ having minimal crossing number. Our goal is to construct a diagram D' for the distant union $K_1 \sqcup \ldots \sqcup K_n$ such that $c(D') \leq 152\, c(D)$. (The *distant union* of oriented knots $K_1, \ldots, K_n$, denoted $K_1 \sqcup \ldots \sqcup K_n$, is constructed by starting with n disjoint 3-balls in the 3-sphere, and for $i = 1, \ldots, n$, placing a copy of K_i in the ith ball.) Theorem 1.1 is then a consequence of the following easy lemma which is proved in Section 2.

Lemma 2.1. *Let $K_1 \sqcup \ldots \sqcup K_n$ be the distant union of oriented knots $K_1, \ldots, K_n$. Then*

$$c(K_1 \sqcup \ldots \sqcup K_n) = c(K_1) + \ldots + c(K_n).$$

So, the key to the proof of Theorem 1.1 is to construct the diagram D' with $c(D') \leq 152\, c(D)$. Let X be the exterior of $K = K_1 \sharp \ldots \sharp K_n$. Arising from the connected sum construction of K, there is a collection of n disjoint annuli $A_1, \ldots, A_n$ properly embedded in X. These are shown in Figure 1. Let A be $A_1 \cup \ldots \cup A_n$.

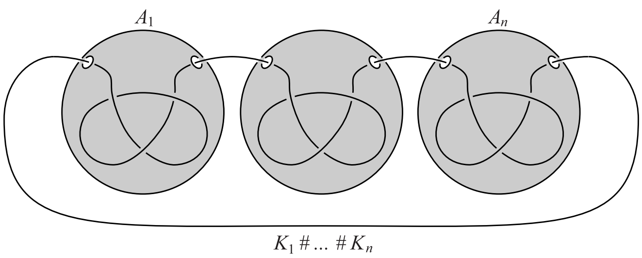

Figure 1.

If one were to cut X along A, the resulting 3-manifold would be the disjoint union of 3-manifolds $X_1, \ldots, X_n$ and Y, where each X_i is homeomorphic to the exterior of K_i, and Y is the component with a copy of each of $A_1, \ldots, A_n$ in its boundary. Each X_i is separated from the other components by a 2-sphere, which is made up of A_i and two meridian discs for K. Thus, if one were to choose on the boundary of each X_i a simple closed curve C_i that intersects the meridian of K_i just once, then the union of these simple closed curves would be $K_1 \sqcup \ldots \sqcup K_n$.

Now, the given diagram D for $K_1 \sharp \ldots \sharp K_n$ need not look like that shown in Figure 1. So, the annuli A may sit inside S^3 in a complicated way. Hence, the 3-manifolds $X_1, \ldots, X_n$ may be embedded in S^3 in a highly twisted fashion, which means that a *priori* the diagram D' of $K_1 \sqcup \ldots \sqcup K_n$ obtained by projecting $C_1 \cup \ldots \cup C_n$ may be very complex. In particular, it may have many more crossings than D. The goal is to gain enough control over the annuli A and hence over the manifolds $X_1, \ldots, X_n$, so that we can bound the number of crossings in D'. For this, our main tool will be normal surface theory.

We first construct a handle structure for X, arising from the diagram D. Since the annuli are essential in X, they may be placed in normal form with respect to this handle structure. This alone does not give us enough control, since the annuli may run through each handle many times. However, if a handle contains many normal discs, they fall into a bounded number of disc types so that any two discs of the same type are normally parallel. Our main technical achievement is to show that the curves C_i can be chosen so that they miss the normal discs that have parallel copies on both sides. So, they run through each handle a bounded number of times and in a controlled way. Thus, we can bound the number of new crossings that are introduced when constructing D' from D, to obtain the inequality $c(D') \leq 152\, c(D)$. The constant 152 arises from the combinatorics of normal discs in our chosen handle structure.

The paper is organised as follows. In Section 2, we will prove Lemma 2.1, which gives the formula for $c(K_1 \sqcup \ldots \sqcup K_n)$. In Section 3, we assign a handle structure to X using the diagram D for K. In Section 4, we recall some of the theory of normal surfaces in handle structures. (For a more complete reference, see [5].) Once we have placed the annuli A into normal form in X, we cut X along A and discard the component Y that contains a copy of A in its boundary. The resulting 3-manifold M is a disjoint union of $X_1, \ldots, X_n$. It inherits a handle structure. Let S be the copy of A in ∂M. In Section 5, we define the notion of a generalised parallelity bundle in M. This is a subset $\mathcal{B}$ of M homeomorphic to an I-bundle over a surface, such that the ∂I-bundle is $\mathcal{B} \cap S$, and which has various other properties. An example of a generalised parallelity bundle is the union of all the handles of M that lie between parallel normal discs of A. In Section 5, we establish the existence of a generalised parallelity bundle $\mathcal{B}$ that contains all these handles, and maybe others, and which has the following key property: *either* the handle structure of M admits a certain type of simplification, known as an annular simplification (in which case, we perform this simplification and continue) *or* each component of $\mathcal{B}$ is an I-bundle over a disc or has incompressible vertical boundary. (The vertical boundary of an I-bundle is the closure of the subset of the boundary that does not lie in the ∂I-bundle; it is a collection of annuli.) In Section 6, we complete the proof of Theorem 1.1. We note that M does not admit any essential embedded annuli with boundary lying in S, because we may assume that the knots $K_1, \ldots, K_n$ are prime. Thus, with further work, we deduce that the generalised parallelity bundle $\mathcal{B}$ is a collection of I-bundles over discs. Hence, its ∂I-bundle does not separate the boundary components of each component of S. It is therefore possible to choose the curves $C_1, \dots, C_n$ so that they avoid $\mathcal{B}$. In particular, they avoid the handles of M that lie between parallel normal discs of A. They therefore they run through each handle of X a bounded number of times and in a controlled way. The final parts of Section 6 are devoted to quantifying this control, and justifying the constant 152.

2. The crossing number of the distant union of knots

In this short section, we prove the following lemma, which is a key step in the proof of Theorem 1.1.

Lemma 2.1. *Let $K_1 \sqcup \dots \sqcup K_n$ be the distant union of oriented knots $K_1, \dots, K_n$. Then*

$$c(K_1 \sqcup \dots \sqcup K_n) = c(K_1) + \dots + c(K_n).$$

Proof. To prove the inequality $c(K_1 \sqcup \dots \sqcup K_n) \geq c(K_1) + \dots + c(K_n)$, consider a diagram D of $K_1 \sqcup \dots \sqcup K_n$ with minimal crossing number. From this, one can construct a diagram D_i of K_i, by eliminating all other components. Thus, $c(D_i)$ is the number of crossings of D where the over-arc and under-arc both lie in K_i. The sum $\sum_{i=1}^n c(D_i)$ therefore enumerates a subset of the crossings of D. Hence,

$$c(K_1 \sqcup \dots \sqcup K_n) = c(D) \geq \sum_{i=1}^{n} c(D_i) \geq \sum_{i=1}^{n} c(K_i).$$

The inequality in the other direction is trivial, since one can construct a diagram for $K_1 \sqcup \dots \sqcup K_n$ from minimal crossing number diagrams of $K_1, \dots, K_n$. □

It is intriguing that distant unions are so much more tractable than connected sums.

3. A handle structure from a diagram

In this section, we describe a method for constructing a handle structure $\mathcal{H}$ on the exterior of a knot K, starting with a diagram D for K. In some sense, it not particularly important how one does this. As long as one picks a handle structure in a reasonably sensible way, then the remaining techniques in this paper will give a result like Theorem 1.1, but possibly with 152 replaced by a different constant.

The diagram D is a 4-valent graph embedded in a 2-sphere S^2, with crossing information at each vertex. Associated with D, there are two collections of disjoint arcs embedded in the 2-sphere, which we denote by D_+ and D_-. Roughly speaking, D_+ is the collection of arcs made by the pen when one draws the knot. That is, one makes two small cuts near each vertex of D, so that the over-arc runs smoothly through the crossing, but the under-arcs are terminated. The resulting collection of arcs is D_+. The arcs D_- are defined similarly, but where the over-arcs are cut at each crossing and the under-arcs run through smoothly.

We realise the 3-sphere as the set of points (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with Euclidean norm 2, say. We embed the diagram 2-sphere as the equator $\{x_4 = 0\}$. The north and south poles of S^3 are the points $(0, 0, 0, 2)$ and $(0, 0, 0, -2)$ respectively. There is a homeomorphism from the complement of these two points to $S^2 \times (-2, 2)$, such that projection $S^2 \times (-2, 2) \to (-2, 2)$ onto the second factor of the product agrees with the height function x_4.

The diagram D specifies an embedding of K into the 3-sphere, as follows. Away from a small regular neighbourhood of the crossings, the knot lies in the diagram 2-sphere. Near each crossing, the knot leaves this 2-sphere, forming two arcs, one lying above the diagram, and one below it. Specifically, the over-arc runs vertically up from the diagram, then runs horizontally at height $x_4 = 1$ say, and then goes vertically back down to the diagram. The under-arc has a similar itinerary below the diagram. Thus, the diagrammatic projection map from the complement of the north and south poles onto the diagram 2-sphere is the product projection map $S^2 \times (-2, 2) \to S^2$.

We pick a point ∞ in the diagram 2-sphere S^2 that is distant from the crossings, and assign a Euclidean metric to $S^2 - \{\infty\}$.

We now define a handle structure $\mathcal{H}'$ on the exterior of K. The handle structure $\mathcal{H}$ that we actually use in the proof of Theorem 1.1 will be a slight modification of this.

We start with the 0-handles of $\mathcal{H}'$. Near each crossing, we place four 0-handles, as shown in Figure 2. Instead of using round 3-balls for the 0-handles, it is slightly more convenient to take each to be of the form $D^2 \times [-1, 1]$, where D^2 is a Euclidean disc, and the second factor is the x_4 co-ordinate.

We now add the 1-handles. Near each crossing, we add four 1-handles. These are 'horizontal', in the sense that they are regular neighbourhoods of arcs in the diagram 2-sphere. These four 1-handles run between the four 0-handles like the edges of a square. Note that these 1-handles do indeed lie in the exterior of K because K skirts above and below the diagram 2-sphere at these points. In addition, for each edge of the 4-valent graph of the knot projection, we add two horizontal 1-handles, which lie either side of the edge and run parallel to it.

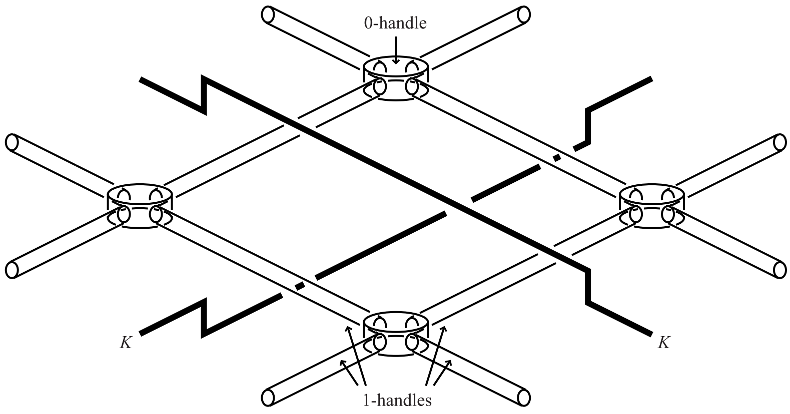

Figure 2.

We now specify where the 2-handles lie. A square-shaped 2-handle, as shown in Figure 3, is attached to the square-shaped configuration of 1-handles and 0-handles near each crossing. It is 'horizontal', in the sense that it is a thin regular neighbourhood of a subset of the diagram 2-sphere. Thus, it is attached to the 1-handles and 0-handles in the 'plane of the diagram'.

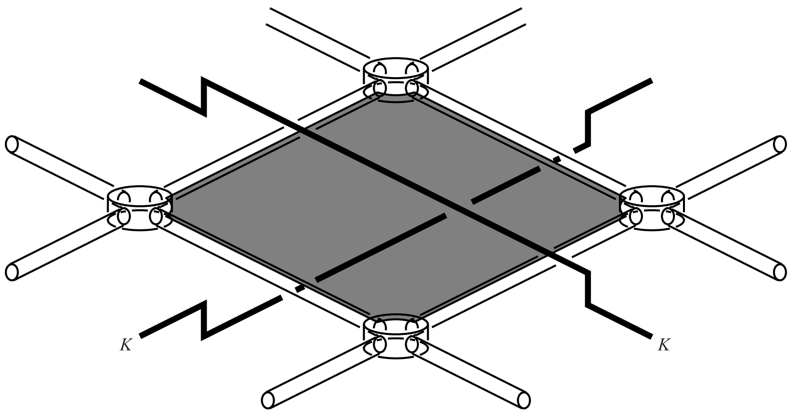

Figure 3.

Associated with each region of the diagram, there is also a horizontal 2-handle. Its attaching annulus runs along the 1-handles and 0-handles that lie within that region, as in Figure 4. It too is attached to the 1-handles and 0-handles in the 'plane of the diagram'.

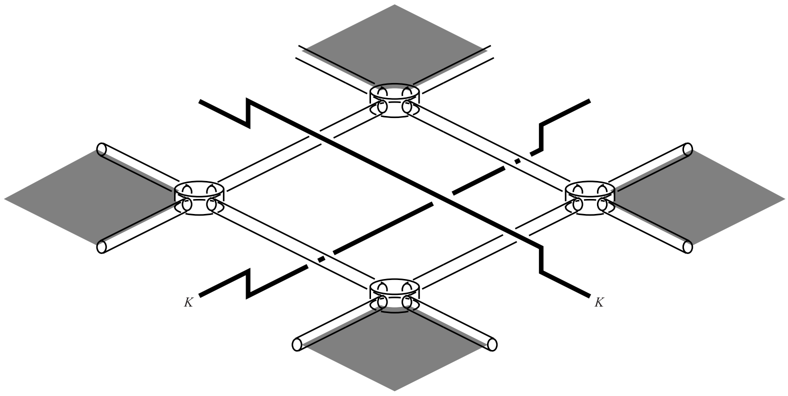

Figure 4.

There is one final type of 2-handle. Each is associated with an arc of D_+ or D_-. It is attached along the 1-handles and 0-handles that encircle this arc, and the 2-handle itself lies over the diagram (in the case of D_+) or under the diagram (in the case of D_-). Part of such a 2-handle is shown in Figure 5. We may suppose that the points where the 2-handle is attached to the 0-handles and 1-handles lie just above the diagram (in the case of D_+) or just below the diagram (in the case of D_-).

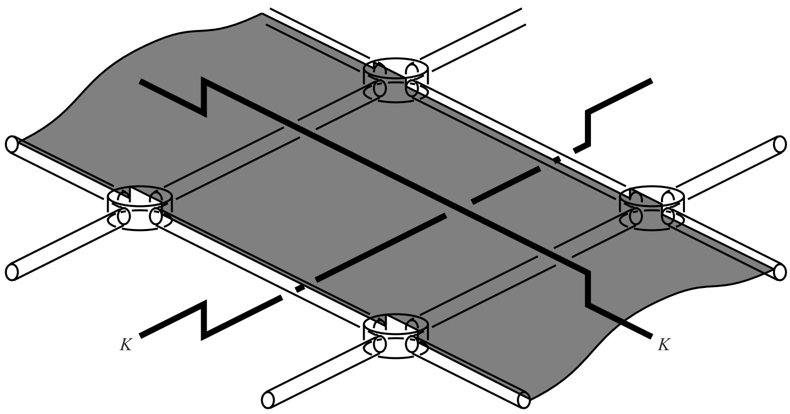

Figure 5.

Finally, there are two 3-handles, one being the 3-ball that lies above all the handles we have just described, the other being the 3-ball that lies below these handles. Thus, we have defined the handle structure $\mathcal{H}'$. Its underlying space is clearly the exterior of K.

We need to specify slightly more precisely how the 2-handles run over the 0-handles and 1-handles. Consider a 0-handle $H_0' = D^2 \times [-1, 1]$ of $\mathcal{H}'$. Its intersection with the 1-handles and 2-handles is the regular neighbourhood of a graph, where the intersection with the 1-handles is a collection of thickened vertices, and the intersection with the

2-handles forms thickened edges. The four thickened vertices are arranged cyclically around the annulus $\partial D^2 \times [-1,1]$. Thus, we may speak of two thickened vertices in $\partial H_0'$ as being *opposite* or *adjacent*. We may arrange that each thickened edge that runs between adjacent thickened vertices is a thickening of an arc $\beta \times p$, where β is an arc in ∂D^2 and p is a point of $[-1,1]$. We may also arrange that each thickened edge running between opposite vertices intersects $\partial D^2 \times [-1,1]$ in two thickened vertical arcs, and intersects $D^2 \times \{-1,1\}$ in a thickened Euclidean geodesic. Thus, it is not hard to see that the way that the 1-handles and 2-handles of $\mathcal{H}'$ are attached to H_0' is as shown in Figure 6. In fact, this specific arrangement is that of the 0-handle at the bottom of Figures 2 - 5.

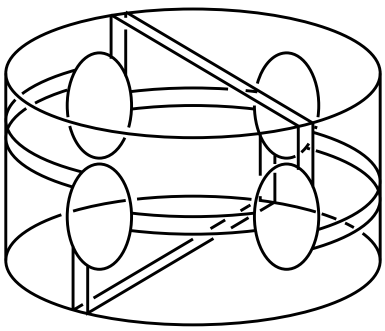

Figure 6.

When the precise embedding of the 0-handle H_0' in S^3 is immaterial, we will usually distort the above picture so that the intersection between H_0' and the 1-handles and 2-handles is planar, as shown in Figure 7.

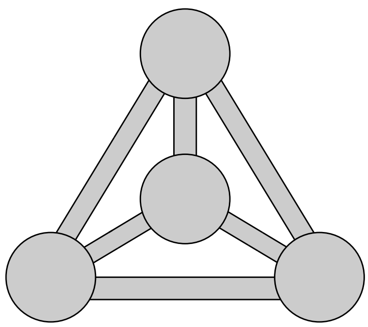

Figure 7.

We now modify $\mathcal{H}'$ slightly to give the handle structure $\mathcal{H}$. Pick a point on K away from the crossings. Running either side of this point, parallel to K, are two 1-handles. Subdivide these, by introducing a 0-handle into each. Above and below the knot at this point, there are two 2-handles. Subdivide each of these, by introducing a 1-handle into each, which runs between the two new 0-handles. (See Figure 8.) Now remove one of these newly introduced 2-handles above K, cancelling it with the 3-handle that lies above the diagram. Do the same with the 2-handle directly below it. (See Figure 8.)

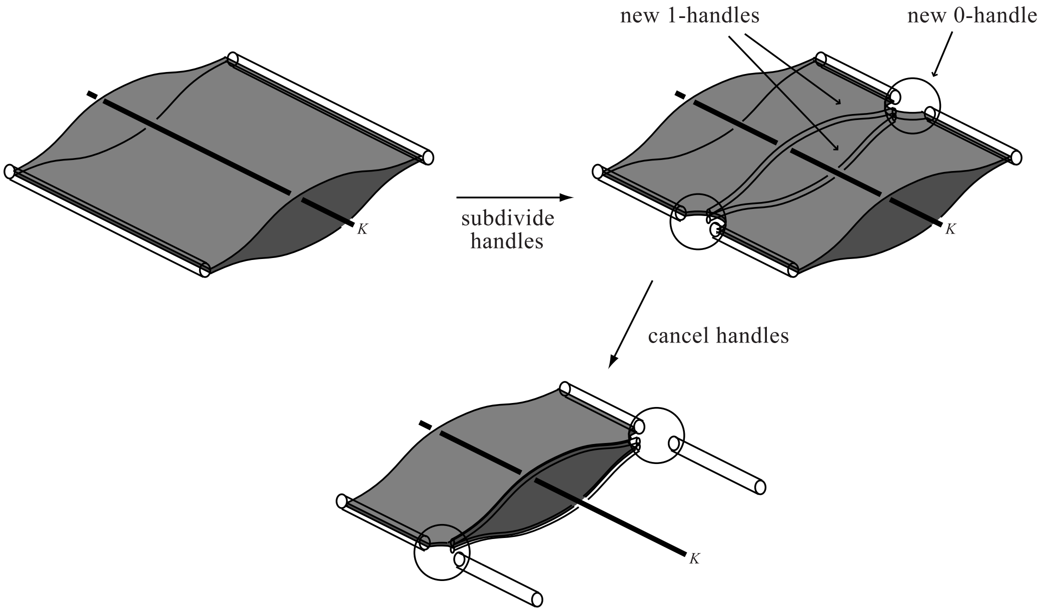

Figure 8.

Let $\mathcal{H}$ be the resulting handle structure, which we call a *diagrammatic handle structure*. We term the two new 0-handles of $\mathcal{H}$ that do not lie in $\mathcal{H}'$ as *exceptional*. The remaining 0-handles are *unexceptional*. For each $i \in \{0, 1, 2, 3\}$, let $\mathcal{H}^i$ be the union of the i-handles of $\mathcal{H}$. Let $\mathcal{F}$ be the surface $\partial\mathcal{H}^0 \cap (\mathcal{H}^1 \cup \mathcal{H}^2)$. As above, this surface may be viewed as the regular neighbourhood of a graph, with the thickened vertices (denoted $\mathcal{F}^0$) being $\mathcal{H}^0 \cap \mathcal{H}^1$ and the thickened edges (denoted $\mathcal{F}^1$) being $\mathcal{H}^0 \cap \mathcal{H}^2$. Note that when H_0 is an unexceptional 0-handle of $\mathcal{H}$, then $H_0 \cap \mathcal{F}$ is as shown in Figures 6 and 7, but possibly with some thickened edges removed.

4. Normal surfaces in handle structures

We now have a diagrammatic handle structure $\mathcal{H}$ on the exterior of the knot K arising from the diagram D. When K is a connected sum $K_1 \sharp \dots \sharp K_n$ of non-trivial knots $K_1, \dots, K_n$, recall that there are associated annuli $A_1, \dots, A_n$ properly embedded in the exterior of K. Let A be their union. A key step in our argument is to place A into normal form with respect to $\mathcal{H}$. In this section, we recall what is meant by normal surfaces in a handle structure $\mathcal{H}$ on a compact 3-manifold X. As in Section 3, we denote the union of the i-handles of a handle structure $\mathcal{H}$ by $\mathcal{H}^i$.

Convention 4.1. We will insist throughout this paper that any handle structure on a 3-manifold satisfies the following conditions:

(i) each i-handle $D^i \times D^{3-i}$ intersects $\bigcup_{j \leq i-1} \mathcal{H}^j$ in $\partial D^i \times D^{3-i}$;

(ii) any two i-handles are disjoint;

(iii) the intersection of any 1-handle $D^1 \times D^2$ with any 2-handle $D^2 \times D^1$ is of the form $D^1 \times \alpha$ in $D^1 \times D^2$, where α is a collection of arcs in ∂D^2, and of the form $\beta \times D^1$ in $D^2 \times D^1$, where β is a collection of arcs in ∂D^2;

(iv) each 2-handle of $\mathcal{H}$ runs over at least one 1-handle.

The diagrammatic handle structure constructed in Section 3 satisfies these requirements.

Let $\mathcal{F}$ be the surface $\mathcal{H}^0 \cap (\mathcal{H}^1 \cup \mathcal{H}^2)$, let $\mathcal{F}^0$ be $\mathcal{H}^0 \cap \mathcal{H}^1$, and let $\mathcal{F}^1$ be $\mathcal{H}^0 \cap \mathcal{H}^2$. By the above conditions, $\mathcal{F}$ is a thickened graph, where the thickened vertices are $\mathcal{F}^0$ and the thickened edges are $\mathcal{F}^1$.

Definition 4.2. We say that a surface A properly embedded in X is *standard* if

(i) it intersects each 0-handle in a collection of properly embedded disjoint discs;

(ii) it intersects each 1-handle $D^1 \times D^2$ in $D^1 \times \beta$, where β is a collection of properly embedded disjoint arcs in D^2;

(iii) it intersects each 2-handle $D^2 \times D^1$ in $D^2 \times P$, where P is a collection of points in the interior of D^1;

(iv) it is disjoint from the 3-handles.

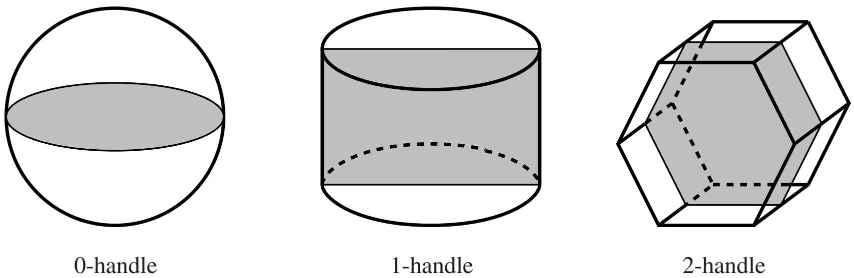

Figure 9.

A standard surface A is termed *normal* if its intersection with the 0-handles satisfies some conditions, as follows.

Definition 4.3. A disc component D of $A \cap \mathcal{H}^0$ is said to be *normal* if

(i) ∂D intersects any thickened edge of $\mathcal{F}$ in at most one arc;

(ii) ∂D intersects any component of $\partial \mathcal{F}^0 - \mathcal{F}^1$ at most once;

(iii) ∂D intersects any component of $\partial \mathcal{H}^0 - \mathcal{F}$ in at most one arc and no simple closed curves.

A standard surface that intersects each 0-handle in a disjoint union of normal discs is said to be *normal*. (See Figure 10.)

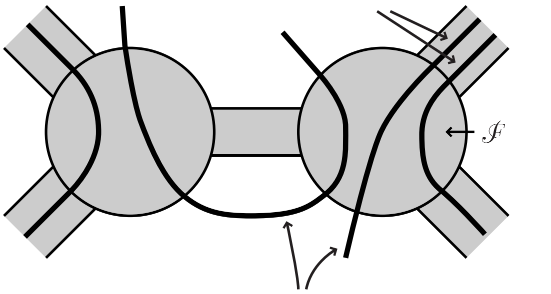

Figure 10.

This is a slightly weaker version of normality than is used by some authors, for example Definition 3.4.1 in [5]. However, if we had used the definition in [5], Proposition 4.4 (below) would no longer have held.

If A is a normal surface in X, then we also say that a component of intersection between A and a 1-handle or 2-handle of $\mathcal{H}$ is a *normal disc*.

Let H be a handle of $\mathcal{H}$. Then two normal discs D and D' in H are *normally isotopic* if there an ambient isotopy, preserving each handle of $\mathcal{H}$, taking D to D'. The discs are then said to be of the same *normal disc type*. It is a standard fact in normal surface theory that, for each handle H in a handle structure, there is an upper bound on the number of normal disc types in H, and these disc types are all constructible. (See p.140 in [5] for example.) Indeed, when H_0 is an unexceptional 0-handle of the diagrammatic handle structure and D is a normal disc in H_0 that is disjoint from ∂X, then D runs over three or four thickened edges, forming either a *triangle* or *square*, as in Figure 11. However, there is a multitude of different normal disc types which intersect ∂X.

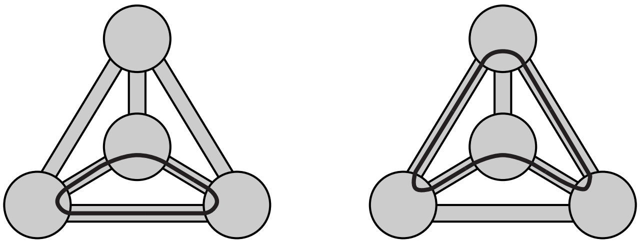

Figure 11.

We now introduce a notion of standard curves in the boundary of a 3-manifold. Let $\mathcal{H}$ be a handle structure on a compact 3-manifold X. Then a collection of disjoint simple closed curves in ∂X is *standard* if

(i) it is disjoint from the 2-handles;

(ii) it intersects each 1-handle $D^1 \times D^2$ in $D^1 \times P$, where P is a collection of points in ∂D^2;

(iii) it intersects $\mathrm{cl}(\partial \mathcal{H}^0 - \mathcal{F})$ in a collection of properly embedded arcs.

Note that when A is a normal surface in a handle structure $\mathcal{H}$ of a 3-manifold X, the manifold M obtained by cutting X along A inherits a handle structure $\mathcal{H}'$. Note moreover that the copies of ∂A in ∂M are standard simple closed curves in $\mathcal{H}'$.

We now wish to place the annuli A into normal form in the handle structure $\mathcal{H}$ on X, the exterior of K. We first ambient isotope ∂A so that it runs over the two exceptional 0-handles and the two 1-handles that run between them, as shown in Figure 12. Note that ∂A is then standard in ∂X. Recall that A divides X into 3-manifolds $X_1, \dots, X_n$ and Y, where each X_i is a copy of the exterior of K_i. We may also ensure that for each $i \in \{1, \dots, n\}$, $\partial X_i \cap \partial X$ also lies in the exceptional 0-handles and the 1-handles that run between them.

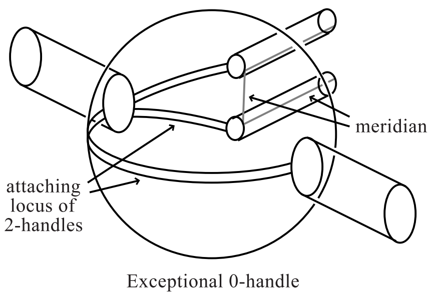

Figure 12.

We now apply the following proposition, which is a slight variant of a well-known result in normal surface theory.

Proposition 4.4. *Let $\mathcal{H}$ be a handle structure on a compact irreducible 3-manifold X. Let A be a properly embedded, incompressible, boundary-incompressible surface in X, with no 2-sphere components. Suppose that each component of ∂A is standard in ∂X and intersects each component of $\partial X \cap \mathcal{H}^0$ and $\partial X \cap \mathcal{H}^1$ in at most one arc and no simple closed curves. Then there is an ambient isotopy, supported in the interior of X, taking A into normal form.*

Proof. This is fairly routine, and follows the proof of Theorem 3.4.7 in [5] for example. So, we will only sketch the argument. We may first ambient isotope A so that it misses the 3-handles of $\mathcal{H}$ and respects the product structure on the 1-handles and 2-handles. Suppose that some component of intersection between A and a 0-handle H_0 is not a disc. Then $A \cap H_0$ is compressible, via a compression disc D_1. Since A is incompressible, ∂D_1 bounds a disc D_2 in A. As X is irreducible, $D_1 \cup D_2$ bounds a 3-ball, and we may ambient isotope D_2 onto D_1. This either reduces $|A \cap \mathcal{H}^2|$, or it leaves it unchanged and reduces $|A \cap \mathcal{H}^1|$. So, we may assume that A intersects each 0-handle in a collection of discs. If a component of intersection between A and a 1-handle is not a disc, then it is an annulus, which forms part of a 2-sphere component of A, contrary to assumption. Thus, A is now standard. Consider a component D of $A \cap \mathcal{H}^0$. If this intersects some thickened edge of $\mathcal{F}$ more than once, then there is an ambient isotopy, which reduces $|A \cap \mathcal{H}^2|$. So, we may assume that (i) in Definition 4.3 (the definition of normality) holds. Suppose that ∂D intersects a component of $\partial \mathcal{H}^0 - \mathcal{F}$ in more than one arc. These two arcs may be joined by an arc α in $\partial \mathcal{H}^0 - \mathcal{F}$. The endpoints of α may be joined by a properly embedded arc β in D. By choosing D suitably, we may ensure that the interior of α is disjoint from A. Then $\alpha \cup \beta$ bounds a disc D' in $\mathcal{H}_0$ such that $D' \cap \partial \mathcal{H}^0 = \alpha$ and $D' \cap A = \beta$. Now, A is boundary-incompressible, and so β separates A into two components, one of which is a disc. In particular, the endpoints of α lie in the same component of ∂A. Hence, this component of ∂A intersects a component of $\partial X \cap \mathcal{H}^0$ in more than one arc, which is contrary to hypothesis. Also, ∂D cannot intersect $\partial \mathcal{H}^0 - \mathcal{F}$ in a simple closed curve, by hypothesis. Thus, (iii) in Definition 4.3 is verified. Finally, ∂D cannot intersect $\partial \mathcal{F}^0 - \mathcal{F}^1$ more than once, since this would imply that a component of ∂A runs over a component of $\partial X \cap \mathcal{H}^1$ more than once. Thus, D satisfies (ii) of Definition 4.3, and so A is normal. □

5. Generalised parallelity bundles

Recall that we are going to cut the exterior of K along the annuli A. The result will be the disjoint union of 3-manifolds $X_1, \dots, X_n$ and Y, where each X_i is homeomorphic to the exterior of K_i. We will choose on the boundary of each X_i (after $X_1 \cup \dots \cup X_n$ has been ambient isotoped) a simple closed curve C_i which hits a meridian of K_i just once. Then, the diagram D' for $K_1 \sqcup \dots \sqcup K_n$ will be the projection of $C_1 \cup \dots \cup C_n$. Our goal is to restrict the number of crossings of D'.

Now, the boundary of $X_1 \cup \dots \cup X_n$ is partitioned into two subsurfaces: a copy of A and parts of $\partial N(K)$. Let S be the former surface. The parts of $\partial N(K)$ in $X_1 \cup \dots \cup X_n$ form a reasonably controlled subsurface of S^3. However, the annuli S may be complicated. They consist of normal discs (one for each normal disc of A), but A may be made up of many normal discs. One might hope to prove Theorem 1.1 by bounding the total number of normal discs of A. However, to prove Theorem 1.1 in this way, this bound would need to be *linear* in the number of crossings of D. Now, there are known bounds on the number of normal discs of certain surfaces in handle structures, for example, Lemma 3.2 of [2]. But these are exponential in the number of 0-handles of $\mathcal{H}$. It seems unlikely that one can achieve a linear bound in general. Thus, a new approach is required. Suppose that A intersects a handle of $\mathcal{H}$ in many normal discs. Then many of these must be normally parallel. The region between two adjacent parallel normal discs of A is a product $D \times I$, where $(D \times I) \cap A = D \times \partial I$. These parallelity regions combine to form I-bundles embedded in the exterior of A. In this section, we consider a generalisation of this structure, known as a 'generalised parallelity bundle'.

Let M be a compact orientable 3-manifold with a handle structure $\mathcal{H}$, and let S be a subsurface of ∂M such that ∂S is standardly embedded in ∂M. We then say that $\mathcal{H}$ is a *handle structure* for the pair (M, S). The main example we will consider is where $M = X_1 \cup \dots \cup X_n$, and S is the copy of A in M.

Definition 5.1. A handle H of $\mathcal{H}$ is a *parallelity handle* if it admits a product structure $D^2 \times I$ such that

(i) $D^2 \times \partial I = H \cap S$;

(ii) each component of $\mathcal{F}^0 \cap H$ and $\mathcal{F}^1 \cap H$ is $\beta \times I$, for a subset β of ∂D^2.

We will typically view the product structure $D^2 \times I$ as an I-bundle over D^2.

The main example of a parallelity handle arises when M is obtained by cutting a 3-manifold X along a normal surface A, and where S is the copies of A in M. Then, if A contains two normal discs in a handle that are normally parallel and adjacent, the space between them becomes a parallelity handle in M. See Figure 13.

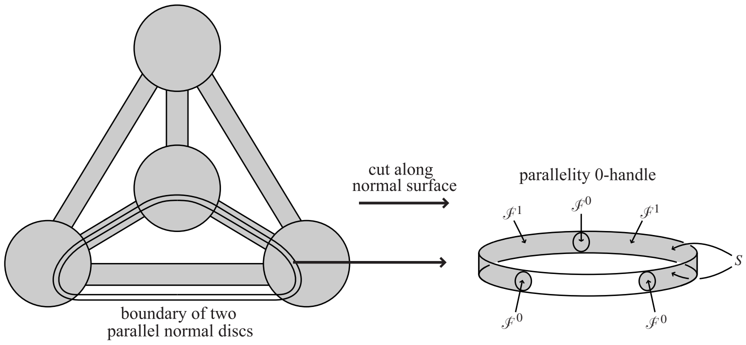

Figure 13.

We now collate some facts about parallelity handles.

A 3-handle can never be a parallelity handle because 3-handles are disjoint from ∂M, and hence disjoint from S.

A 2-handle $D^2 \times D^1$ is a parallelity handle if and only if $(D^2 \times D^1) \cap S = D^2 \times \partial D^1$. In this case, the two product structures that the handle has, one from the fact that it is a parallelity handle, the other from the fact that is a 2-handle, can be made to coincide.

A 1-handle $D^1 \times D^2$ is a parallelity handle if and only if $(D^1 \times D^2) \cap S$ is two discs and each component of $(D^1 \times \partial D^2) - S$ lies entirely in ∂M or entirely in $\mathcal{H}^2$. In this case, the I-bundle structure on the 1-handle can be made to respect its structure as a product $D^1 \times D^2$, in the sense that D^2 inherits a structure as $I \times I$, so that fibres in the I-bundle are of the form $p_1 \times p_2 \times I$, for $p_1 \in D^1$ and $p_2 \in I$.

When two parallelity handles are incident, we will see that their I-bundle structures can be made to coincide along their intersection. So, the union of the parallelity handles forms an I-bundle over a surface F, say. (See Lemma 5.3.) It will be technically convenient to consider enlargements of such structures. These will still be an I-bundle over a surface F, and near the I-bundle over ∂F, they will be a union of parallelity handles, but elsewhere need not be. The precise definition is as follows.

Definition 5.2. Let $\mathcal{H}$ be a handle structure for the pair (M, S). A *generalised parallelity bundle* $\mathcal{B}$ is a 3-dimensional submanifold of M such that

(i) $\mathcal{B}$ is an I-bundle over a compact surface F;

(ii) the ∂I-bundle is $\mathcal{B} \cap S$;

(iii) $\mathcal{B}$ is a union of handles of $\mathcal{H}$;

(iv) any handle in $\mathcal{B}$ that intersects the I-bundle over ∂F is a parallelity handle, where I-bundle structure on the parallelity handle agrees with the I-bundle structure of $\mathcal{B}$;

(v) $\mathrm{cl}(M - \mathcal{B})$ inherits a handle structure.

The I-bundle over ∂F is termed the *vertical boundary* of $\mathcal{B}$, and the ∂I-bundle over F is called the *horizontal boundary*.

Note that a single 2-handle $D^2 \times D^1$ such that $(D^2 \times D^1) \cap S = D^2 \times \partial D^1$ is a generalised parallelity bundle. An example of a slightly more complicated generalised parallelity bundle is shown in Figure 14. It is composed of two parallelity 2-handles and a parallelity 1-handle.

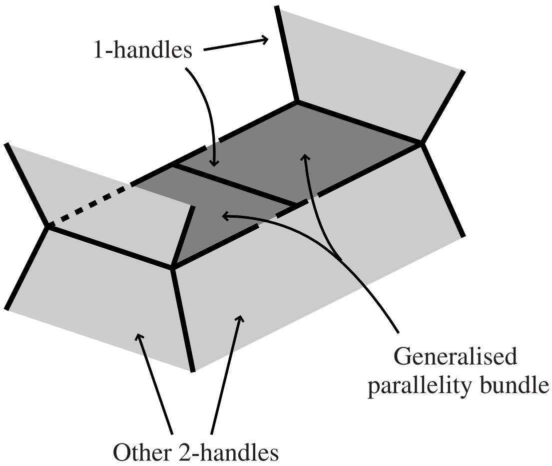

Figure 14.

The point of the definition is that generalised parallelity bundles behave in many ways like 2-handles, in that the remaining handles form a handle structure, onto which the generalised parallelity bundle is attached. However, note that the vertical boundary of a generalised parallelity bundle need not be properly embedded in M. This is because the vertical boundary of a parallelity handle may intersect ∂M in its interior. But, the intersection between the vertical boundary and ∂M is a union of fibres in the I-bundle.

The following lemma gives an important example of a generalised parallelity bundle.

Lemma 5.3. *The union of the parallelity handles is a generalised parallelity bundle.*

Proof. By definition, each parallelity handle has the structure of an I-bundle. We claim that these structures can be chosen so that they coincide on the intersection of any two parallelity handles. Hence, the union $\mathcal{B}$ of the parallelity handles will inherit an I-bundle structure.

Consider first the intersection of a parallelity 0-handle H_0 and a parallelity 1-handle H_1. These intersect along components of $\mathcal{F}^0$. By condition (ii) in the Definition 5.1 (the definition of a parallelity handle), each such component of $\mathcal{F}^0$ inherits a product structure $\beta_0 \times I$ from H_0 and a product structure $\beta_1 \times I$ from H_1. By condition (i) in Definition 5.1 applied twice,

$$\beta_0 \times \partial I = (\beta_0 \times I) \cap S = (\beta_1 \times I) \cap S = \beta_1 \times \partial I.$$

Hence, these product structures can be made to coincide. A similar argument applies to the intersection of a parallelity 0-handle and a parallelity 2-handle, but with the role of $\mathcal{F}^0$ replaced by $\mathcal{F}^1$. Finally consider the intersection of a parallelity 1-handle H_1 and a parallelity 2-handle H_2. Now, $H_1 \cap \mathcal{F}^1$ is a disjoint union of fibres, by condition (ii) in the Definition 5.1. Thus, the I-bundle structures of H_1 and H_2 agree along $H_1 \cap H_2 \cap \mathcal{H}^0$. Since the I-bundle structures respect the product structures on H_1 and H_2, we see that they agree along all of $H_1 \cap H_2$.

Thus, conditions (i) - (iv) in Definition 5.2 (the definition of a generalised parallelity bundle) follow immediately. We must check condition (v) in Definition 5.2, which asserts that $\mathrm{cl}(M - \mathcal{B})$ inherits a handle structure. The only way that this might fail is if a j-handle of $\mathrm{cl}(M - \mathcal{B})$ is incident to an i-handle of $\mathcal{B}$, for $j > i$. Thus, we must check that if an i-handle is a parallelity handle, then so is any j-handle to which it is incident, for $j > i$. Let us consider when the i-handle is a 0-handle H_0. Then, by definition of the parallelity structure on H_0, each component of $H_0 \cap \mathcal{F}^0$ and $H_0 \cap \mathcal{F}^1$ inherits an I-bundle structure, which therefore extends over any incident 1-handle or 2-handle, making it a parallelity handle. Note also that a parallelity 0-handle is not incident to any 3-handles. So, the claim holds for $i = 0$. Let us now consider the case where $i = 1$, and let $H_1 = D^1 \times D^2$ be a parallelity 1-handle. Then $H_1 \cap S$ is two discs, and each component of $(D^1 \times \partial D^2) - S$ lies entirely in ∂M or entirely in $\mathcal{H}^2$. Thus, H_1 is disjoint from the 3-handles. Also, any 2-handle $D^2 \times D^1$ to which it is incident has both components of $D^2 \times \partial D^1$ lying in S. So, it is a parallelity handle, as required. Finally, the case where $i = 2$ follows from the observation that a parallelity 2-handle is disjoint from the 3-handles. This proves the claim. It is now clear that conditions (i) - (iv) in Convention 4.1 hold for $\mathrm{cl}(M - \mathcal{B})$, and hence it inherits a handle structure. □

Suppose that M is irreducible and S is incompressible. Our aim now is to construct a handle structure on (M, S) containing a generalised parallelity bundle that satisfies the following two conditions:

(i) it contains every parallelity handle;

(ii) its horizontal boundary is incompressible in M.

This will be achieved via the following procedure for simplifying a handle structure $\mathcal{H}$ of (M, S).

Definition 5.4. Let G be an annulus properly embedded in M, with boundary in S. Suppose that there is an annulus G' in ∂M such that $\partial G = \partial G'$. Suppose also that $G \cup G'$ bounds a 3-manifold P such that

(i) either P is a parallelity region between G and G', or P lies in a 3-ball;

(ii) P is a non-empty union of handles;

(iii) $\mathrm{cl}(M - P)$ inherits a handle structure from $\mathcal{H}$;

(iv) any parallelity handle of $\mathcal{H}$ that intersects P lies in P;

(v) G is a vertical boundary component of a generalised parallelity bundle lying in P;

(vi) $G' \cap (\partial M - S)$ is either empty or a regular neighbourhood of a core curve of the annulus G'.

Removing the interiors of P and G' from M is called an *annular simplification*. Note that the resulting 3-manifold M' is homeomorphic to M, even though P may be homeomorphic to the exterior of a non-trivial knot when it lies in a 3-ball. (See Figure 15.) The boundary of M' inherits a copy of S, which we denote by S', as follows. We set $S' \cap \partial M$ to be $S \cap \partial M'$. When $G' \cap (\partial M - S)$ is empty, we declare that $\partial M' - \partial M$ lies in S'. When $G' \cap (\partial M - S)$ is a single annulus, we declare that $\partial M' - \partial M$ is disjoint from S'. Thus, (M', S') is homeomorphic to (M, S). Moreover, when M is embedded within a bigger closed 3-manifold, then (M', S') is ambient isotopic to (M, S).

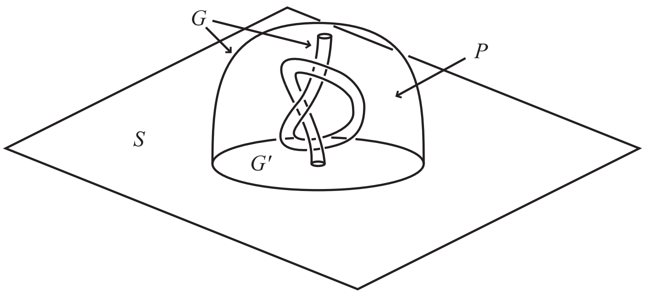

Figure 15.

Lemma 5.5. *Let $\mathcal{H}$ be a handle structure for the pair (M, S). Let $\mathcal{H}'$ be a handle structure obtained from $\mathcal{H}$ by an annular simplification. Then any parallelity handle for $\mathcal{H}$ that lies in $\mathcal{H}'$ is a parallelity handle for $\mathcal{H}'$.*

Proof. Let H be a parallelity handle for $\mathcal{H}$. Let P be the 3-manifold, the interior of which is removed in the annular simplification. If H intersects P, then it lies in P by condition (iv) in Definition 5.4 (the definition of an annular simplification), and hence it does not lie in $\mathcal{H}'$. So, if H lies in $\mathcal{H}'$, then it does not intersect P, and so it was not modified in the annular simplification. Thus, it is a parallelity handle for $\mathcal{H}'$. □

A generalised parallelity bundle is *maximal* if it is not strictly contained within another generalised parallelity bundle. (Thus, it is maximal with respect to the partial order of inclusion, where the inclusion does not necessarily respect the bundle structures.) Note that if a generalised parallelity bundle $\mathcal{B}$ is maximal, then any parallelity handle that intersects $\mathcal{B}$ must lie in $\mathcal{B}$.

Proposition 5.6. *Let M be a compact orientable irreducible 3-manifold with a handle structure $\mathcal{H}$. Let S be an incompressible subsurface of ∂M, such that ∂S is standard in ∂M. Suppose that $\mathcal{H}$ admits no annular simplification. Let $\mathcal{B}$ be any maximal generalised parallelity bundle in $\mathcal{H}$. Then the horizontal boundary of $\mathcal{B}$ is incompressible.*

Proof. Let $\mathcal{B}'$ be those components of $\mathcal{B}$ that are not I-bundles over discs. It clearly suffices to show that the horizontal boundary of $\mathcal{B}'$ is incompressible.

We claim that it suffices to show that the vertical boundary of $\mathcal{B}'$ is incompressible. For, if the vertical boundary were incompressible, then any compression disc for the horizontal boundary could be isotoped off the vertical boundary. Hence, it would lie entirely in the generalised parallelity bundle. But the horizontal boundary of an I-bundle is incompressible in the I-bundle. Thus, the horizontal boundary of $\mathcal{B}'$ is incompressible if the vertical boundary is.

Consider therefore a compression disc D for the vertical boundary of $\mathcal{B}'$. Let V be the vertical boundary component containing ∂D. By the definition of $\mathcal{B}'$, D does not lie entirely in $\mathcal{B}'$. Its interior is disjoint from $\mathcal{B}'$ (by the definition of a compression disc), but it may intersect $\mathcal{B} - \mathcal{B}'$. Note that V is properly embedded in M, since the interior of D lies on one side of it, and a component of $\mathcal{B}'$ lies on the other side.

Now, V compresses along D to give two discs D_1' and D_2' embedded in M, with boundary in S. Since S is incompressible and M is irreducible, D_1' and D_2' are parallel to discs D_1 and D_2 in S, via 3-balls P_1 and P_2. There are two cases to consider: where P_1 and P_2 are disjoint and where they are nested.

Let us suppose first that they are disjoint. Then, $V \cup D_1 \cup D_2$ bounds a 3-ball B. Since the interior of D is disjoint from $\mathcal{B}'$, this ball B does not lie in $\mathcal{B}'$. So, we may extend the I-bundle structure of $\mathcal{B} - B$ over B, contradicting the maximality of $\mathcal{B}$. See Figure 16. Note that, here, we are using the fact that generalised parallelity bundles need not consist solely of parallelity handles.

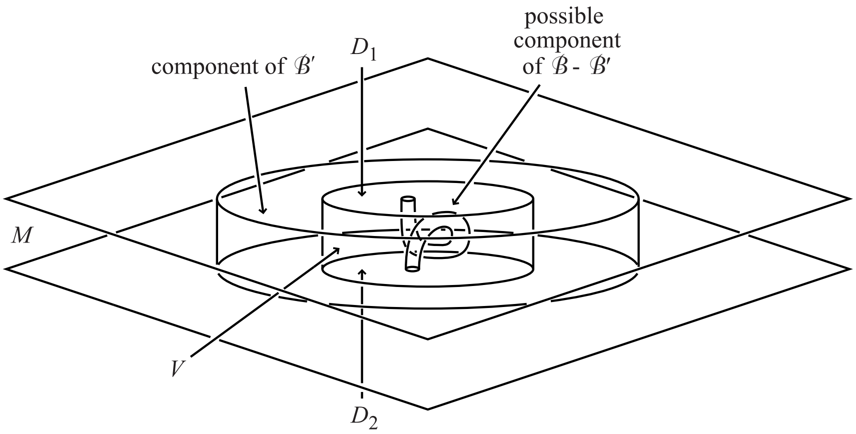

Figure 16.

Let us now suppose that P_1 and P_2 are nested; say that P_2 lies in P_1. See Figure 17. Let G' be $D_1 - \mathrm{int}(D_2)$. Then, G' is an annulus in S such that $\partial V = \partial G'$. Let P be the 3-manifold bounded by $V \cup G'$. This lies in the 3-ball P_1. By (v) in Definition 5.2 (the definition of a generalised parallelity bundle), $\mathrm{cl}(M - P)$ inherits a handle structure. Note also that, by the maximality of $\mathcal{B}$, any parallelity handle that intersects P lies in P. So, $\mathcal{H}$ admits an annular simplification, which is a contradiction. □

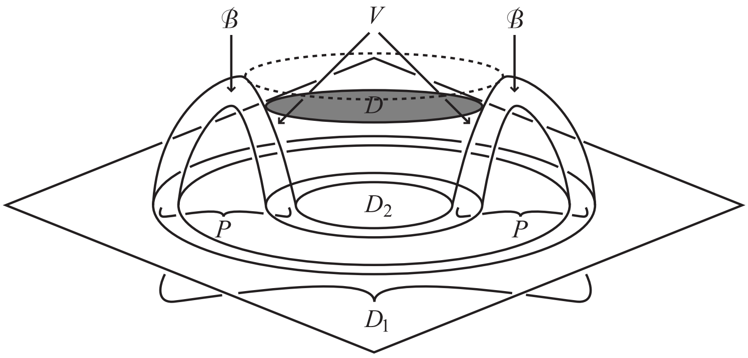

Figure 17.

Corollary 5.7. *Let M, S and $\mathcal{H}$ be as in Proposition 5.6. Then there is a generalised parallelity bundle $\mathcal{B}$ such that*

(i) $\mathcal{B}$ contains every parallelity handle;

(ii) the horizontal boundary of $\mathcal{B}$ is incompressible.

Proof. By Lemma 5.3, the union of the parallelity handles is a generalised parallelity bundle. Enlarge this to a maximal generalised parallelity bundle $\mathcal{B}$. By Proposition 5.6, its horizontal boundary is incompressible. □

Proposition 5.8. *Let M be a compact, orientable, irreducible 3-manifold, with boundary a collection of incompressible tori. Let S be a subsurface of ∂M such that the intersection of S with each component of ∂M is either empty or a single incompressible annulus. Suppose that any incompressible annulus properly embedded in M with boundary in S is boundary parallel. Let $\mathcal{H}$ be a handle structure for (M, S) that admits no annular simplifications. Then there is a generalised parallelity bundle $\mathcal{B}$ such that*

(i) $\mathcal{B}$ contains every parallelity handle;

(ii) $\mathcal{B}$ is a collection of I-bundles over discs.

Proof. Let $\mathcal{B}$ be the generalised parallelity bundle provided by Corollary 5.7. Let $\mathcal{B}'$ be the union of the components of $\mathcal{B}$ that are not I-bundles over discs. Its horizontal boundary is a subsurface of S, which is a collection of annuli. The only connected compact incompressible subsurface of an annulus is an annulus or disc. Therefore, each component of $\mathcal{B}'$ is an I-bundle over an annulus or Mobius band. The vertical boundary components of $\mathcal{B}'$ are incompressible annuli, with boundary curves in S. Thus, by assumption, each such vertical boundary component is boundary-parallel in M.

We claim that no component of $\mathcal{B}'$ is an I-bundle over a Mobius band. Let V be the vertical boundary of such a component B of $\mathcal{B}'$. Now, V is boundary parallel in M, via a parallelity region P. The interior of P is disjoint from B, since B is an I-bundle over a Mobius band. Hence, M is homeomorphic to B, which is a solid torus. But, this is a contradiction, because we have assumed that ∂M is incompressible. This proves the claim.

Consider a component V of the vertical boundary of $\mathcal{B}'$, and let P be the parallelity region between V and an annulus in ∂M. The component of $\mathcal{B}'$ containing V is an I-bundle over an annulus, and so its vertical boundary components are parallel. So, by changing the choice of V if necessary, we may assume that P contains this component of $\mathcal{B}'$. Then V must be properly embedded in M, for otherwise one could find a compression disc for ∂M in P. Hence, if one were to remove the interiors of P and $\partial P - V$ from $\mathcal{H}$, this would be an annular simplification, contrary to hypothesis. So, $\mathcal{B}'$ must be empty, and therefore $\mathcal{B}$ is a collection of I-bundles over discs, as required. □

6. Proof of the main theorem

In this section, we will complete the proof of Theorem 1.1. Suppose that K is a connected sum of oriented knots $K_1, \dots, K_n$.

6.1. We may assume that each K_i is prime and non-trivial

Express each K_i as a connected sum of prime knots $K_{i,1}, \dots, K_{i,m(i)}$. Suppose that we could prove the theorem in the case where each summand is prime. Then we would have the inequality

$$c(K) \geq \frac{\sum_{i=1}^{n} \sum_{j=1}^{m(i)} c(K_{i,j})}{152}.$$

But the trivial inequality for connected sums gives that

$$\sum_{j=1}^{m(i)} c(K_{i,j}) \geq c(K_i),$$

and so this would imply that

$$c(K) \geq \frac{\sum_{i=1}^{n} c(K_i)}{152}.$$

Thus, it suffices to consider the case where each K_i is prime. We may also clearly assume that each K_i is non-trivial.

6.2. Handle structures and normal surfaces

Let D be a diagram of K with minimal crossing number. Our aim is to construct a diagram D' for the distant union $K_1 \sqcup \dots \sqcup K_n$, such that $c(D') \leq 152\, c(D)$. This will prove the theorem.

Let X be the exterior of K. Give X the diagrammatic handle structure described in Section 3. Recall that the expression of K as a connected sum $K_1 \sharp \dots \sharp K_n$ specifies a collection of annuli $A_1, \dots, A_n$ properly embedded in X, as shown in Figure 1. Let A be their union. We first perform the isotopy of ∂A that is described in Section 4 just before Proposition 4.4. It then intersects only the exceptional 0-handles and the 1-handles that run between them. (See Figure 12.) Then, using Proposition 4.4, we ambient isotope A, keeping ∂A fixed, taking it to a normal surface.

Let $X_1 \cup \dots \cup X_n \cup Y$ be the result of cutting X along A. Let M be $X_1 \cup \dots \cup X_n$ and let S be the copy of A in M. Let $\mathcal{H}$ be the handle structure that M inherits. Note that ∂S is standard in ∂M. Thus, $\mathcal{H}$ is a handle structure for the pair (M, S).

6.3. Applying annular simplifications

We claim that all of $\partial M \cap \partial X$ lies in the parallelity handles of $\mathcal{H}$. Note that there is only one possibility, up to normal isotopy, for the normal discs of A in the exceptional 0-handles. (See Figure 18.) They are therefore normally parallel. Moreover, in the isotopy of ∂A in Section 4, we arranged that $\partial M \cap \partial X$ lies in the exceptional 0-handles and the 1-handles that run between them. The claim now follows.

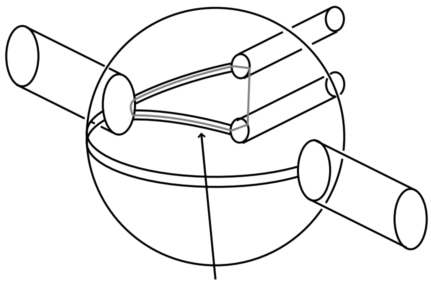

Figure 18.

Let R be the union of the parallelity handles in M, and let R' be the union of the components of R that are incident to ∂X.

We apply as many annular simplifications to $\mathcal{H}$ as possible, giving a handle structure $\mathcal{H}'$ on a pair (M', S') ambient isotopic to (M, S). Denote the components of M' by $X'_1, \dots, X'_n$, and let $S'_i = S' \cap \partial X'_i$. Thus, $S' = S'_1 \cup \dots \cup S'_n$.

By Lemma 5.5, any parallelity handle of $\mathcal{H}$ that lies in M' is a parallelity handle for $\mathcal{H}'$. Note that, because each K_i is prime, any incompressible annulus properly embedded in M' with boundary in S' is boundary parallel. Also, $\partial M'$ is incompressible, since each K_i is non-trivial. Hence, by Proposition 5.8, $\mathcal{H}'$ has a generalised parallelity bundle $\mathcal{B}$ that contains all parallelity handles of $\mathcal{H}'$ and that consists of I-bundles over discs.

We claim that all of R' was removed when constructing $\mathcal{H}'$ from $\mathcal{H}$. For, in each annular simplification, each component of R is either entirely removed or left untouched, by condition (iv) in Definition 5.4 (the definition of an annular simplification). Hence, each component of R is either a subset of $\mathcal{B}$ or was removed. But each component of R' contains a component of $\partial M \cap \partial X$ and so cannot lie in an I-bundle over a disc. Thus, each component of R' was removed in the annular simplifications, as claimed. Recall that, in this situation, we declared that $\mathrm{cl}(\partial M' - S')$ is a collection of vertical boundary components of generalised parallelity bundles that are removed. In particular, $\mathrm{cl}(\partial M' - S')$ inherits the structure of an I-bundle. Now, the vertical boundary of $\mathcal{B}$ lies in a collection of parallelity handles of $\mathcal{H}'$, by (iv) in Definition 5.2 (the definition of a generalised parallelity bundle). None of these handles is incident to $\partial M' - S'$, for this would have violated condition (iv) in Definition 5.4 (the definition of an annular simplification). So, the vertical boundary of $\mathcal{B}$ is disjoint from $\partial M' - S'$.

6.4. Choosing the curves $C_1, \dots, C_n$

Recall that we are going to pick a simple closed curve C_i on the boundary of each X_i'. The union of these curves will be $K_1 \sqcup \dots \sqcup K_n$, and their projection will be the diagram D'. We first declare that $C_i \cap \mathrm{cl}(\partial M' - S')$ is a fibre in the I-bundle structure on $\mathrm{cl}(\partial M' - S')$. We may also arrange that this arc lies in a 0-handle of $\mathcal{H}'$. The remainder of C_i will be an arc α_i in S_i' joining the endpoints of this fibre. We will choose the arcs α_i so as to control the crossing number of the resulting diagram D'.

Let $\alpha = \alpha_1 \cup \dots \cup \alpha_n$. Our first task is to ensure that α sits nicely with respect to a certain handle structure on S', which we now define. Now, S' consists of two parts:

(i) copies of normal discs of A, which we denote by S_A';

(ii) vertical boundary components of generalised parallelity bundles that have been removed by annular simplifications, which we denote by S_V'.

Thus, $S' = S_A' \cup S_V'$. The normal discs in A specify a handle structure on S_A', where (for $j = 0, 1, 2$) the j-handles are the intersection with $\mathcal{H}^j$. We may extend this to a handle structure on all of S', by declaring that the intersection of S_V' with $\mathcal{H}^0$ is 1-handles (running between the two boundary components of the relevant vertical annulus), and that the remainder of S_V' is 2-handles.

We may ambient isotope α within S', keeping $\partial\alpha$ fixed, so that it misses the 2-handles of S' and respects the product structure on the 1-handles. This implies that in a 1-handle $H_1 = D^1 \times D^2$ of the diagrammatic handle structure, $\alpha \cap H_1$ is of the form $D^1 \times P$, where P is a disjoint union of points in the interior of D^2. We may also arrange that the restriction of the diagrammatic projection map to $\alpha \cap H_1$ is an embedding. (Recall that the 1-handles of the diagrammatic handle structure are 'horizontal' in S^3.)

Now, some handles of S' lie in the generalised parallelity bundle $\mathcal{B}$. But, crucially, $\mathcal{B}$ is a collection of I-bundles over discs, disjoint from $\partial M' - S'$. Thus, the intersection $S' \cap \mathcal{B}$, which is the horizontal boundary of $\mathcal{B}$, is a collection of discs in the interior of S'. We may therefore pick the arcs α_i so that they avoid the generalised parallelity bundle $\mathcal{B}$, without changing the choice of $\partial\alpha_i$.

Define the *length* of α to be the number of 0-handles of S' that it runs through (with multiplicity). We pick α so that it has shortest possible length among arcs that avoid the generalised parallelity bundle $\mathcal{B}$. Hence, for each 0-handle D of S', $\alpha \cap D$ is at most one arc. Otherwise, we may find an embedded arc β in D such that $\alpha \cap \beta = \partial\beta$ and such that the endpoints of β lie in distinct components of $D \cap \alpha$. Cut α along $\partial\beta$, discard the arc that misses $\partial S'$ and replace it by β. The result is a shorter collection of arcs than α, which is a contradiction.

The fact that α misses the generalised parallelity bundle $\mathcal{B}$ implies, in particular, that it misses the parallelity regions in $X'_1 \cup \ldots \cup X'_n$ between parallel normal discs of A. Thus, in any handle of the diagrammatic handle structure, α can run over at most two normal discs of any given disc type. For, in any collection of three or more parallel normal discs, all but the outer two discs have parallelity regions on both sides. Suppose that α runs over a disc D that is not an outer disc. Then D lies in $X'_1 \cup \ldots \cup X'_n$, because α does. Moreover, one of the parallelity regions adjacent to D is a parallelity handle of $\mathcal{H}'$ and so lies in $\mathcal{B}$. But α misses $\mathcal{B}$, which is a contradiction.

We now wish to find an upper bound on the crossing number of the diagram D'. In order to do this, we need to be precise about how the normal discs of A lie in each 0-handle, and how α intersects these discs.

6.5. The position of the normal discs of A

The normal discs of A come in two types: those that miss the boundary of X, and those that intersect ∂X. The discs of the latter type have been removed when creating $\mathcal{H}'$. Thus, the normal discs of A that lie in S' all miss ∂X, and are therefore triangles and squares in unexceptional 0-handles, as shown in Figure 11. Each unexceptional 0-handle of the diagrammatic handle structure simultaneously supports at most four triangle types and at most one type of square.

We say that a normal disc E properly embedded in a 0-handle of the diagrammatic handle structure is *flat* if, for each point x in the diagram 2-sphere, the inverse image of x in E under the projection map is either empty, a single point, or an arc in ∂E. We say that E is *semi-flat* if it contains a properly embedded arc δ such that the closure of each component of $E - \delta$ is flat. We say that a flat disc is *convex* if the image of its projection is a convex subset of the diagram 2-sphere. (Recall that we have assigned a Euclidean metric to $S^2 - \{\infty\}$, where S^2 is the diagram 2-sphere and ∞ is a point that is distant from the crossings, and hence the 0-handles.) We say that a semi-flat disc is *piecewise-convex* if its two flat subdiscs are convex. It is clear that we can make all of the triangles and squares of A simultaneously flat and convex, apart from certain squares, as shown in Figure 19, which can be made semi-flat and piecewise convex.

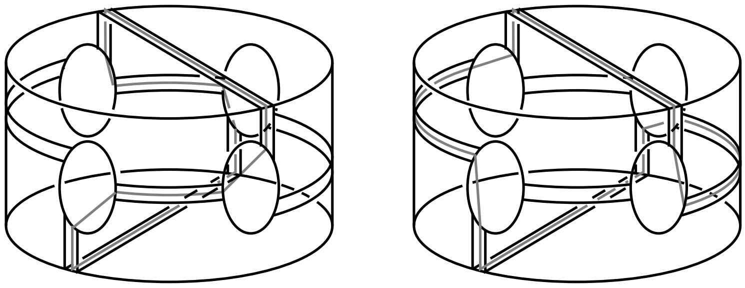

Figure 19.

When a triangle or square E is flat and convex, realise $\alpha \cap E$ as the unique arc in E that projects to a Euclidean geodesic under the diagrammatic projection map and that has the same endpoints. When E is semi-flat and piecewise convex, it contains a properly embedded arc δ such that each component of $E - \delta$ is flat and convex. We may therefore realise the intersection of α with each such component as an arc that projects to Euclidean geodesic. So, $\alpha \cap E$ projects in this case to the concatenation of two Euclidean geodesics. We call such an arc *bent*.

Thus, we have defined the precise location of the simple closed curves C. Its projection is the diagram D'. We now wish to bound the crossing number of D'.

6.6. Justifying the constant 152

The arcs $C \cap (\partial M' - S')$ are the interiors of fibres in the parallelity regions between adjacent normal discs of A. Thus, we may clearly arrange that their projections in D' have disjoint images and are disjoint from the image of $C \cap S'$. The crossings of D' therefore arise when arcs of intersection between C and distinct normal discs of A do not have disjoint projections. The crossings occur only within the projections of the unexceptional 0-handles. Consider one such 0-handle H_0. The intersection $C \cap H_0$ lies in at most 10 normal discs, of which all but at most 2 are flat and convex. The non-flat discs are semi-flat and piecewise convex. Its intersection with each flat disc projects to straight arc. The intersection with each semi-flat disc projects to a bent arc. Now, the projection of two straight arcs has at most one crossing. There are at most $\binom{8}{2} = 28$ such crossings. The projection of two bent arcs has at most 4 crossings. The projection of a bent arc and a straight arc has at most 2 crossings, and there are therefore at most 32 such crossings. So, the number of crossings of α in the projection of H_0 is at most $28 + 4 + 32 = 64$.

In fact, we may improve this estimate a little. For each type of triangle in H_0, there is another triangle type in H_0, with the property that the projections of these triangles to the diagram are disjoint. Hence, we may reduce the upper bound on the number of straight-straight crossings by 8 to 20. We also note that the projection of the arc δ in each semi-flat disc can each be arranged to lie in the 'centre' of the projection of H_0. More specifically, we can define the central region to be the intersection of the image of the two horizontal pieces of $\mathcal{F}^1 \cap H_0$, and we can ensure that δ is a vertical arc that projects to this central region. Thus, the two geodesics in each bent arc are nearly radial. This implies that the projection of each such geodesic is disjoint from the projection of two triangle types. Hence, the upper bound on the number of straight-bent crossings can be reduced by 16 to 16. Finally, by carefully arranging the semi-flat discs, we may ensure that if there are two bent arcs, then they intersect at most twice. So, the number of crossings of C in the projection of H_0 is at most $20 + 2 + 16 = 38$. An example of the projection of H_0 containing 38 crossings is given in Figure 20.

The number of unexceptional 0-handles is $4\ c(D)$. Thus, $c(D')$ is at most $152\ c(D)$, which proves Theorem 1.1. □

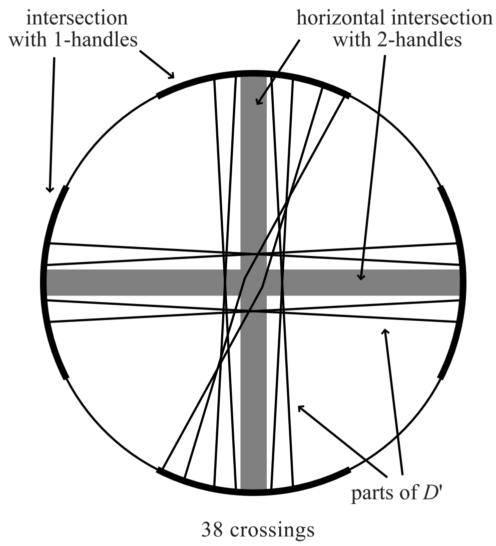

Figure 20.

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Mathematical Institute, University of Oxford,
24-29 St Giles', Oxford OX1 3LB, United Kingdom.