The Geometry and Topology of Finite Index Subgroups

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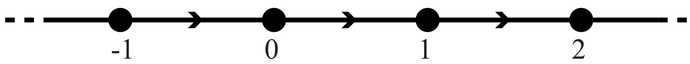
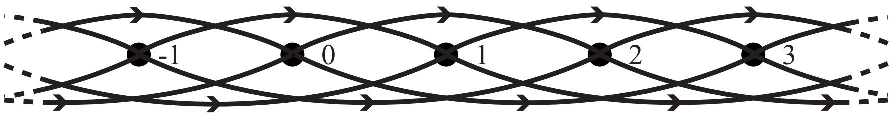
Let Γ be a group with finite generating set S.

Let $X(\Gamma; S)$ = the Cayley graph of Γ with respect to S.
Its vertices: elements of Γ
Its edges: $g \leftrightarrow gs$ ($g \in \Gamma$, $s \in S \cup S^{-1}$)
Make each edge have length 1.

This makes $X(\Gamma; S)$ and Γ into metric spaces, but the precise metric depends on the choice of generating set S.

$X(\mathbb{Z}; \{1\})$ 

$X(\mathbb{Z}; \{2, 3\})$ 

Quasi-isometry

If S and S' are two different finite generating sets for Γ, then $X(\Gamma; S)$ and $X(\Gamma; S')$ are quasi-isometric.

A quasi-isometry between two metric spaces (M, d) and (M', d') is a function $f: M \to M'$ for which there exists a constant $K \geq 1$ such that, for all $x_1, x_2 \in M$,

1. $$\frac{d(x_1, x_2)}{K} - K \leq d'(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + K$$

2. $$N_K(f(M)) = M'.$$

f is known as a K-quasi-isometry.

Quasi-isometry induces an equivalence relation on metric spaces.

Proof

Give Γ a metric d_S arising as a subspace of $X(\Gamma; S)$. Then the inclusion $\Gamma \to X(\Gamma; S)$ is a 1-quasi-isometry.

So, to prove that $X(\Gamma; S)$ and $X(\Gamma; S')$ are quasi-isometric, it suffices to supply a quasi-isometry $(\Gamma, d_S) \to (\Gamma, d_{S'})$.

Write each $s \in S$ as a word in the generators S', and each $s' \in S'$ as a word in the generators S. Let K be the maximum length of these words.

Then the identity map is a K-quasi-isometry $(\Gamma, d_S) \to (\Gamma, d_{S'})$.

The aims of geometric group theory

1. Find quasi-isometry invariants of groups
2. Relate them to algebraic properties
3. Link to other areas of mathematics, eg lattices in Lie groups.

A celebrated example: growth of groups

Let $Gr(n)$ be the number of vertices in $X(\Gamma; S)$ with distance at most n from the identity vertex.

This is the growth function of the group. [Milnor]

Obviously, the precise function depends on the choice of S, but the following property depends only Γ:

Γ has polynomial growth if there is a polynomial $p(n)$ such that $Gr(n) \leq p(n)$ for all n.

Theorem: [Gromov] A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Finite index subgroups

There are many interesting and natural questions relating to finite index subgroups.

Some definitions:

$vb_1(\Gamma) = \sup\{b_1(H) : H \text{ is a finite index subgroup of } \Gamma\}$

A group Γ is large if some finite index subgroup admits a surjective homomorphism onto a non-abelian free group.

Note: Large $\Rightarrow vb_1 = \infty$.

Can one detect geometrically if a group Γ has $vb_1 > 0$? Or has $vb_1 = \infty$? Or is large?

Let Γ be the fundamental group of a closed orientable 3-manifold M.

Conjecture: (The virtually Haken conjecture)
If $|\Gamma| = \infty$, then some finite index subgroup Γ_i is an amalgamated free product or HNN extension.

(The covering space M_i of M corr. to Γ_i is known as Haken.)

$$\Uparrow$$

Conjecture: [Thurston - Waldhausen]
If $|\Gamma| = \infty$, then $vb_1(\Gamma) > 0$.

$$\Uparrow$$

$$\Uparrow$$

(assuming the Geometrisation Conjecture)

Conjecture: (The infinite virtual b_1 conjecture)
If Γ is a lattice in $\mathrm{PSL}(2,\mathbb{C})$, then $vb_1(\Gamma) = \infty$.

$$\Uparrow$$

Conjecture: (The largeness conjecture)
If Γ is a lattice in $\mathrm{PSL}(2,\mathbb{C})$, then Γ is large.

However, traditional geometric group theory does not interact well with the theory of finite index subgroups.

Two reasons:

1. [Burger-Moses] There exist quasi-isometric groups Γ and Γ' s.t

(i) Γ is simple (and so has no proper, finite index subgroups); in particular $vb_1(\Gamma) = 0$ and Γ is not large;

(ii) Γ' is residually finite and large.

2. Any finite index subgroup of Γ is quasi-isometric to Γ. So, traditional geometric group theory cannot distinguish a group from any of its finite index subgroups.

Let

Γ = a group

S = a finite generating set

$\{\Gamma_i\}$ = a collection of finite index (normal) subgroups

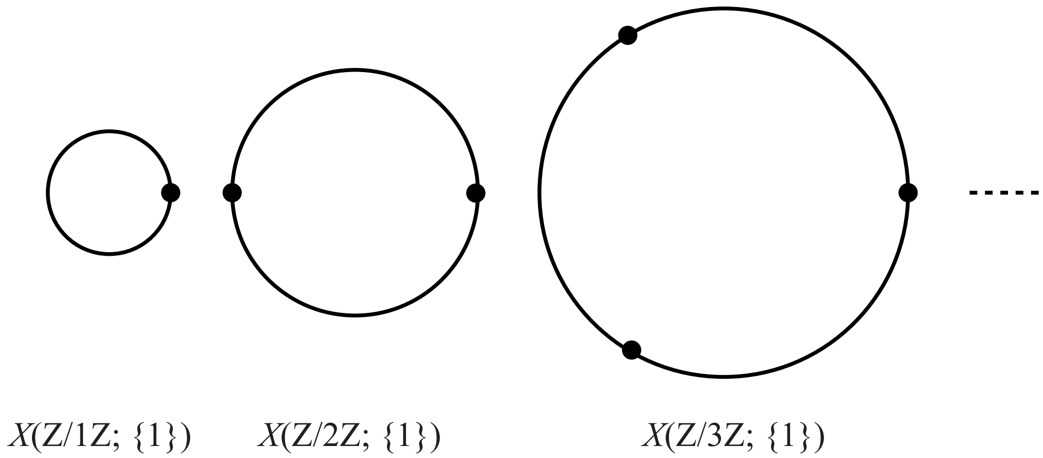
Then S descends to a generating set for Γ/Γ_i.

Consider the metric spaces $X(\Gamma/\Gamma_i; S)$.

The quasi-isometry class of this sequence does not depend on the choice of S.

Two sequences M_i and M_i' of metric spaces are quasi-isometric if there is a $K \geq 1$ and, for each i, a K-quasi-isometry between M_i and M_i'.

eg: $\Gamma = \mathbb{Z}$, $\Gamma_i = i\mathbb{Z}$.



Basic Principle

Algebraic properties of the subgroups Γ_i	$\longleftrightarrow$	Geometry of $X(\Gamma/\Gamma_i; S)$

Examples of geometric properties of Cayley graph $X = X(\Gamma/\Gamma_i; S)$:

Its diameter

Its Cheeger constant $h(X)$

The first eigenvalue of its Laplacian, denoted $\lambda_1(X)$

Examples of algebraic properties of Γ_i:

Amalgamated free product?

Positive b_1?

Free non-abelian quotient?

Its rank (ie minimal number of generators)

The rank of its first homology

Let K be a finite 2-complex (or compact Riemannian manifold) with $\pi_1(K) = \Gamma$.

Let $K_i \to K$ be the covering space corresponding to the subgroup Γ_i.

$$\begin{tikzcd}
\text{Geometric properties of } K_i \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{Algebraic properties of } \pi_1(K_i) = \Gamma_i \\
\text{Geometric properties of } X(\Gamma/\Gamma_i; S) &
\end{tikzcd}$$

An example

Theorem: [L] Let Γ be a finitely presented group. Then TFAE:

1. $vb_1(\Gamma) > 0$;
2. Γ has a sequence of finite index subgroups $\Gamma_1 \geq \Gamma_2 \geq \ldots$, each normal in Γ_1, s.t.
$$\lambda_1(X(\Gamma/\Gamma_i)) = O([\Gamma : \Gamma_i]^{-2}).$$
3. Γ has a sequence of finite index subgroups $\Gamma_1 \geq \Gamma_2 \geq \ldots$, each normal in Γ_1, s.t.
$$\lambda_1(X(\Gamma/\Gamma_i)) = o([\Gamma : \Gamma_i]^{-1}).$$

The Laplacian on graphs

Let X be a connected graph with vertex set V.

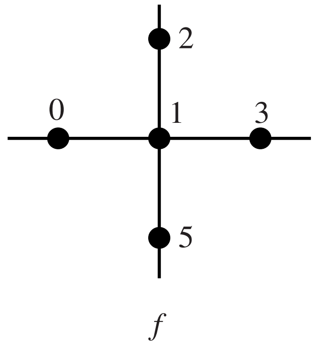
The Laplacian is a linear map

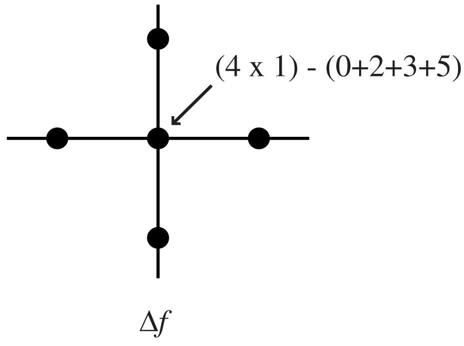
$$\Delta\colon \mathbb{R}^V \to \mathbb{R}^V$$

defined as follows. Given $f \in \mathbb{R}^V$,

$$\Delta f(v) = \mathrm{valence}(v) f(v) - \sum_{v'} f(v')$$

where v' runs over all neighbours of v.



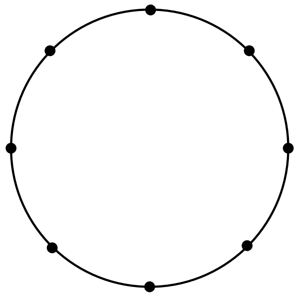


In fact, Δ is self-adjoint with eigenvalue spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

Example

Let X be the circle of n vertices (n even, say).



Then the eigenfunctions of Δ are

$$\theta \mapsto \cos(m\theta) \qquad (0 \leq m \leq n/2)$$

$$\theta \mapsto \sin(m\theta) \qquad (1 \leq m \leq n/2 - 1)$$

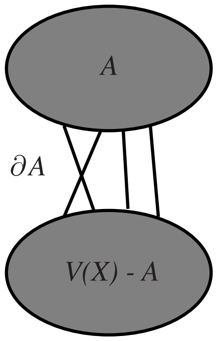
(where $0 \leq \theta < 2\pi$)

The Cheeger constant

Let X be a finite graph.

Its Cheeger constant $h(X)$ is

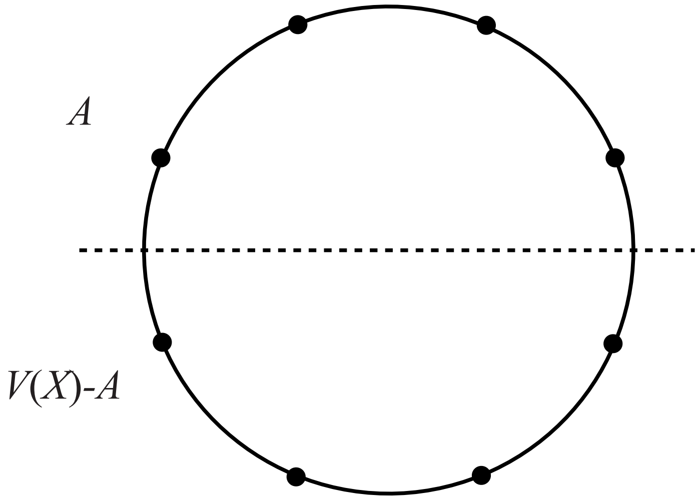
$$\min \left\{ \frac{|\partial A|}{|A|} : \begin{array}{c} A \subset V(X), \\ 0 < |A| \leq |V(X)|/2 \end{array} \right\}.$$



$h(X)$ small $\Leftrightarrow$ there is a 'bottleneck'.

Example

Let $S = \{1\}$. Then $X(\mathbb{Z}/n\mathbb{Z}; S) =$



$$h(X) = \frac{2}{\lfloor n/2 \rfloor}$$

THE RELATIONSHIP BETWEEN h AND λ_1

Thm: [Dodziuk, Alon, Tanner, Alon-Milman]

If d is the maximal valence of any vertex of X, then

$$\frac{h^2(X)}{2d} \leq \lambda_1(X) \leq 2\,h(X)$$

Invariants of Riemannian manifolds

Let

Γ = a group

S = a finite generating set

$\{\Gamma_i\}$ = a collection of finite index (normal) subgroups

M = a closed Riemannian manifold with $\pi_1(M) = \Gamma$

M_i = the covering space corresponding to Γ_i.

One can define $\lambda_1(M_i)$ and $h(M_i)$.

There is a constant K s.t.

$$K^{-1}\,\lambda_1(M_i) \leq \lambda_1(X(\Gamma/\Gamma_i; S)) \leq K\,\lambda_1(M_i)$$

$$K^{-1}\,h(M_i) \leq h(X(\Gamma/\Gamma_i; S)) \leq K\,h(M_i)$$

Continuation of Theorem 1.1.

Theorem 1.2: [L] Let Γ be a finitely presented group. Then TFAE:

1. $vb_1(\Gamma) > 0$;
2. Γ has a sequence of finite index subgroups $\Gamma_1 \geq \Gamma_2 \geq \ldots$, each normal in Γ_1, s.t.

$$h(X(\Gamma/\Gamma_i)) = O([\Gamma : \Gamma_i]^{-1}).$$

3. Γ has a sequence of finite index subgroups $\Gamma_1 \geq \Gamma_2 \geq \ldots$, each normal in Γ_1, s.t.

$$h(X(\Gamma/\Gamma_i)) = o([\Gamma : \Gamma_i]^{-1/2}).$$

Being able to detect $vb_1(\Gamma) > 0$ is interesting (partly because of the link to 3-manifold theory).

But the condition $h(X(\Gamma/\Gamma_i)) = o([\Gamma : \Gamma_i]^{-1/2})$ is very difficult to verify in practice.

In fact, even the following is hard to detect: $h(X(\Gamma/\Gamma_i)) \to 0$.

This relates to Property (τ), which I'll dicsuss next time.

Theorem 1.3: [L] Let Γ be a finitely presented group. Let $\{\Gamma_i\}$ be a nested sequence of finite index normal subgroups. Then one of the following must hold:

1. Γ_i is an amalgamated free product or HNN extension for all $i >> 0$;
2. $\inf_i h(X(\Gamma/\Gamma_i; S)) > 0$;
3. $\lim_{i\to\infty}(d(\Gamma_i) - 1)/[\Gamma : \Gamma_i] = 0$.

Here, $d(\,\cdot\,)$ is the rank of a group, ie the minimal number of generators.

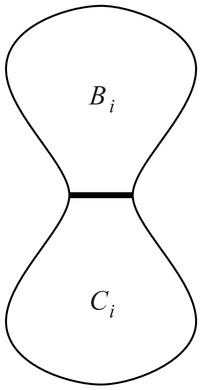
(1) and (3) are algebraic properties of Γ_i

(2) is a geometric property of the Cayley graphs $X(\Gamma/\Gamma_i; S)$.

Idea of proof

Let K = a finite 2-complex with $\pi_1(K) = \Gamma$
K_i = the covering space corresponding to Γ_i.

Suppose that $h(X(\Gamma/\Gamma_i; S))$ is arbitrarily small. Then the K_i have 'bottlenecks'.



This gives a decomposition of K_i into two pieces, suggesting that $\pi_1(K_i)$ is an amalgamated free product.

Rank gradient

The rank gradient of $(\Gamma, \{\Gamma_i\})$ is

$$\inf_i \frac{d(\Gamma_i) - 1}{[\Gamma : \Gamma_i]}.$$

Recall: (Reidermeister-Schreier)

$$d(\Gamma_i) - 1 \leq [\Gamma : \Gamma_i](d(\Gamma) - 1).$$

So, the rank gradient measures how far this is from being an equality.

Examples:

1. $\Gamma = \mathbb{Z}$, $\Gamma_i = i\mathbb{Z}$

Zero rank gradient

2. $\Gamma = \mathbb{Z} * \mathbb{Z}$, $G_i =$ any sequence of f.i. subgroups

Positive rank gradient.

3. $\Gamma = A \rtimes \mathbb{Z}$, where A is finitely generated.

Note $d(\Gamma) \leq d(A) + 1$.

There is a hm $\phi\colon \Gamma \to \mathbb{Z}$. Let $\Gamma_i = \phi^{-1}(i\mathbb{Z})$. Then $\Gamma_i = A \rtimes \mathbb{Z}$.

Zero rank gradient

Proof of theorem

Suppose that $h(X(\Gamma/\Gamma_i)) \to 0$, and
$(\Gamma, \{\Gamma_i\})$ has positive rank gradient.

We will show that Γ_i is an amalgamated free product or HNN extension, for all $i >> 0$.

Replace Γ by some Γ_j, where j is large.

Positive rank gradient $\Rightarrow$ for all $i >> 0$,

$$d(\Gamma_i) \sim [\Gamma : \Gamma_i](d(\Gamma) - 1)$$

Let K = a finite 2-complex with $\pi_1(K) = \Gamma$, with a single 0-cell.

Let K_i = covering space corresponding to Γ_i.

Let X_i = 1-skeleton of K_i.

We're assuming that $h(X_i) \to 0$.

Use this to decompose K_i into two pieces B_i and C_i. Apply Seifert - van Kampen theorem.

Either $\pi_1(K_i)$ is an amalgamated free product or HNN extension

Or $\pi_1(B_i) \twoheadrightarrow \pi_1(K_i)$ or $\pi_1(C_i) \twoheadrightarrow \pi_1(K_i)$, which $\Rightarrow$

$$d(\Gamma_i) = d(\pi_1 K_i) \leq \max\{d(\pi_1 B_i), d(\pi_1 C_i)\} \lesssim (1/2)[\Gamma : \Gamma_i](d(\Gamma) - 1)$$

contradiction. $\square$

Natural questions

How can we ensure that a group Γ has a nested sequence of finite index normal subgroups $\{\Gamma_i\}$

- with positive rank gradient?
- s.t. $h(X(\Gamma/\Gamma_i; S)) \to 0$?

These questions are related!

We'll examine them next time.