Three-dimensional manifolds
Michaelmas Term 1999
Examples Sheet 2

1. Show that any compact non-orientable 3-manifold M having no $\mathbb{R}P^2$ boundary components has infinite $H_1(M)$, and hence has a 2-sided properly embedded non-separating incompressible surface.

2. Construct a Haken 3-manifold with the same homology as S^3.

3. Show that if M is a compact orientable irreducible 3-manifold with $\pi_1(M)$ a free group, then M is a handlebody. [Hint: consider a map from M to a bouquet of circles.]

4. Let M be a compact orientable irreducible 3-manifold and let F be a compact surface in ∂M. Show that if $\pi_1(F) \to \pi_1(M)$ is an isomorphism, then there is a homeomorphism from M to $F \times [0,1]$ taking F to $F \times \{0\}$.

5. Show that if S is an orientable incompressible surface properly embedded in a compact orientable 3-manifold M, then $\pi_1(M_S) \to \pi_1(M)$ is injective.

6. Using a hierarchy argument, show that the fundamental group of a Haken 3-manifold is torsion-free.

7. What extra assumptions do we need to make about a properly embedded incompressible surface with non-empty boundary in a compact irreducible 3-manifold to guarantee that it can be ambient isotoped into normal form?

8. Show that any Haken 3-manifold M has a hierarchy

$$M = M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n,$$

where $n \leq 5$ (but where each surface S_i may be disconnected). Here, n is the *length* of this hierarchy. Which compact orientable 3-manifolds have a hierarchy of length one?

9. Suppose that a compact 3-manifold M contains k properly embedded 2-spheres, none of which bounds a 3-ball and no two of which are parallel. Then show that, for any triangulation of M, we may find such a collection of 2-spheres in normal form. Deduce that any compact 3-manifold can be

expressed as a connected sum of prime 3-manifolds.

10. Show that, in the statement of the Loop Theorem, we may remove the hypotheses that the 3-manifold is irreducible and compact.

11. Disprove the following conjecture. 'Let F be a surface properly embedded in a 3-manifold M, such that, with respect to any triangulation of M, F may be ambient isotoped into normal form. Then F is incompressible.' [Hint: let M be the lens space obtained by $6/1$ surgery on the unknot.]

12. Show that, for a fixed triangulation of a 3-manifold with t tetrahedra, its normal surfaces are in one-one correspondence with the integral lattice points in a subset C of $\mathbb{R}^{7t}$. If S_1, S_2 and S_3 are normal surfaces corresponding to points $[S_1]$, $[S_2]$ and $[S_3]$ in C, such that $[S_1] + [S_2] = [S_3]$, how are the Euler characteristics of S_1, S_2 and S_3 related?